12.2 Techniques for Evaluating Limits

Dividing out Technique

Example: Evaluate $\lim\limits_{x \to 3} \dfrac{x^2 - 9}{x - 3} = \dfrac{9-9}{3-3} = \underbrace{\dfrac{0}{0}}_{\text{indeterminate form}}$

Try factoring: $\lim\limits_{x \to 3} \dfrac{\cancel{(x-3)}(x+3)}{\cancel{(x-3)}} = x+3$

$= \lim\limits_{x \to 3} x+3 = 3+3 = \boxed{6}$

<u>Example:</u> Evaluate $\displaystyle\lim_{x\to -3} \frac{x+3}{x^2+5x+6}$

$$= \lim_{x\to -3} \frac{\cancel{x+3}}{\cancel{(x+3)}(x+2)}$$

$$= \lim_{x\to -3} \frac{1}{x+2} = \frac{1}{-3+2} = \boxed{-1}$$

Rationalizing Technique

Example: Evaluate $\lim_{x\to 0} \frac{\sqrt{x+9}-3}{x} \cdot \frac{(\sqrt{x+9}+3)}{(\sqrt{x+9}+3)}$

Multiply the top and bottom by the conjugate of the numerator: $\sqrt{x+9}+3$

$= \lim_{x\to 0} \frac{x+9-9}{x(\sqrt{x+9}+3)}$ ← don't distribute

$= \lim_{x\to 0} \frac{\cancel{x}}{\cancel{x}(\sqrt{x+9}+3)}$

$= \lim_{x\to 0} \frac{1}{\sqrt{x+9}+3}$

$= \frac{1}{\sqrt{0+9}+3} = \boxed{\frac{1}{6}}$

Example: Evaluate $\lim_{x\to 0} \frac{(\sqrt{x+2}-\sqrt{2})}{x} \cdot \frac{(\sqrt{x+2}+\sqrt{2})}{(\sqrt{x+2}+\sqrt{2})}$

$$= \lim_{x\to 0} \frac{x+2-2}{x(\sqrt{x+2}+\sqrt{2})}$$

$$= \lim_{x\to 0} \frac{x}{x(\sqrt{x+2}+\sqrt{2})}$$

$$= \lim_{x\to 0} \frac{1}{\sqrt{x+2}+\sqrt{2}} = \frac{1}{\sqrt{2}+\sqrt{2}}$$

$$= \frac{1}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \boxed{\frac{\sqrt{2}}{4}}$$

One-Sided Limits

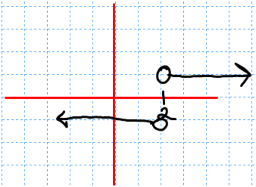
Limit from the left: $\lim_{x \to c^-} f(x) = L$

Limit from the right: $\lim_{x \to c^+} f(x) = L$

$\lim_{x \to c} f(x) = L$ if and only if both the left and the right limits exist and are equal to L.

Example: Find limit of $f(x)$ as x approaches 2.

$$\lim_{x \to 2} \frac{|x-2|}{x-2}$$



$$\lim_{x \to 2^-} \frac{|x-2|}{x-2} = -1$$

$$\lim_{x \to 2^+} \frac{|x-2|}{x-2} = 1$$

$$\lim_{x \to 2^-} \neq \lim_{x \to 2^+}$$

$$-1 \neq 1$$

$$\boxed{\text{DNE}}$$

Example: For $f(x) = 4x + 3$, find

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{[4(x+h)+3] - [4x+3]}{h}$$

$$= \lim_{h\to 0} \frac{4x+4h+3-4x-3}{h}$$

$$= \lim_{h\to 0} \frac{4h}{h} = \lim_{h\to 0} 4 = \boxed{4}$$

<u>Example:</u> For $f(x) = x^2 - x$, find

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{[(x+h)^2-(x+h)] - [x^2-x]}{h}$$

$$= \lim_{h\to 0} \frac{x^2+2xh+h^2-x-h-x^2+x}{h}$$

$$= \lim_{h\to 0} \frac{2xh+h^2-h}{h}$$

$$= \lim_{h\to 0} 2x+h-1 = 2x+0-1 = \boxed{2x-1}$$