9.3 Geometric Sequences and Series

A sequence is geometric if the ratios of consecutive terms are the same. The number *r* is the common ratio.

Examples:

a) 3, 6, 12, 24, …

$3 \xrightarrow{\times 2} 6 \xrightarrow{\times 2} 12 \xrightarrow{\times 2} 24$

$r = \frac{6}{3} = 2$, $r = \frac{12}{6} = 2$

b) $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$

$1 \xrightarrow{\times \frac{1}{2}} \frac{1}{2} \xrightarrow{\times \frac{1}{2}} \frac{1}{4} \rightarrow \frac{1}{8}$

$r = \frac{\frac{1}{2}}{1} = \frac{1}{2}$, $r = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$

The *n*th term of a geometric sequence has the form

$$a_n = a_1 r^{n-1}$$

$a_1 = 1^{st}$ term

$r =$ common ratio , $r = \frac{a_2}{a_1}$

Example: Find the *n*th term for the geometric sequence with first term 5 and common ratio 2.

$a_1 = 5$

$r = 2$

$a_n = a_1 r^{n-1}$

$\boxed{a_n = 5(2)^{n-1}}$

Example: Find the twentieth term of the geometric sequence 1, 3, 9, 27, ...

$a_1 = 1$

$r = \frac{3}{1} = 3$

$n = 20$

$a_{20} = 1(3)^{19}$

$= \boxed{1{,}162{,}261{,}467}$

Example: Find the fifteenth term of the geometric sequence where the third term is $\frac{5}{4}$ and the sixth term is $\frac{5}{32}$.

n	1	....	3	4	5	6
a_n	?		$\frac{5}{4}$			$\frac{5}{32}$

$\times r \quad \times r \quad \times r$

① find a_1, r and a rule.

$$\frac{5}{4} r^3 = \frac{5}{32}$$

$$r^3 = \frac{5}{32} \cdot \frac{4}{5}, \quad r^3 = \frac{1}{8}, \quad r = \sqrt[3]{\frac{1}{8}} \quad = r = \frac{1}{2}$$

② $a_n = a_1 \left(\frac{1}{2}\right)^{n-1}$

use $(3, \frac{5}{4})$ to find a_1:

$$\frac{5}{4} = a_1 \left(\frac{1}{2}\right)^{3-1}$$

$$\frac{5}{4} = a_1 \left(\frac{1}{2}\right)^{2}$$

$$\frac{5}{4} = a_1 \left(\frac{1}{4}\right)$$

$$\left(\frac{4}{1}\right)\frac{5}{4} = a_1$$

$$5 = a_1$$

$\boxed{\text{Rule: } a_n = 5\left(\frac{1}{2}\right)^{n-1}}$

③ $a_{15} = 5\left(\frac{1}{2}\right)^{14} = 3.05 \times 10^{-4} = \boxed{.000305}$

The sum of a finite geometric sequence is

$$S_n = \frac{a_1\left(1-r^n\right)}{1-r}$$

Example: Evaluate $\sum_{n=1}^{20} 2\underbrace{(0.1)}_{r}{}^n$

$a_1 = 2(.1)^1 = .2$

$n = 20$

$$S_{20} = \frac{.2(1-.1^{20})}{1-.1} = \frac{.2(1-.1^{20})}{.9} = \boxed{\frac{2}{9}}$$

The sum of an infinite geometric sequence is

$$S_\infty = \frac{a_1}{1-r} \quad \text{if} \quad |r| < 1$$

Example: Evaluate $\displaystyle\sum_{n=1}^{\infty} 2(\underbrace{0.4}_{r<1\ \checkmark})^{n-1}$

$a_1 = 2(.4)^{1-1}$

$a_1 = 2(.4)^0 = 2$

$S_\infty = \dfrac{2}{1-.4} = \dfrac{2}{.6} = \boxed{\dfrac{10}{3}}$