9.4 Mathematical Induction

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Example: Find P_{k+1} for the following.

a) $P_k = \dfrac{k(k+2)^2}{3}$

replace k with k+1

$$P_{k+1} = \frac{(k+1)\left[(k+1)+2\right]^2}{3} = \frac{(k+1)(k+3)^2}{3}$$

b) $P_k = 1 + 4 + 7 + \cdots + \left[3(k-1)-2\right] + (3k-2)$

$P_{k+1} = 1+4+7+\ldots+\left[3(k+1-1)-2\right] + (3(k+1)-2)$

$P_{k+1} = 1+4+7+\ldots+\left[3k-2\right] + \left[3k+1\right]$

The Principle of Mathematical Induction

Let P_n be a statement involving the positive integer n. If

1. P_1 is true, and
2. anytime P_k is true, P_{k+1} is also true for every positive integer k, then P_n must be true for all positive integers n.

<u>Example:</u> Use mathematical induction to prove the following formula

$$S_n = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

1. Show the formula is true for n = 1.

$$S_1 = \frac{1(1+1)}{2} = \frac{2}{2} = 1$$

2. Two steps:

a) Assume that the formula is valid for some integer *k*. replace n with k:

$$S_k = 1+2+3+\cdots+k = \frac{k(k+1)}{2}$$

b) Use the assumption to prove that the formula is valid for the next integer, *k* + 1.

We need to prove that

$$S_{k+1} = \underbrace{1+2+3+\cdots+k}_{S_k} + k+1 = \frac{(k+1)[(k+1)+1]}{2}$$

Work on the left side to prove the right side

Subst. in from part (a) above

$$\frac{k(k+1)}{2} + k+1 = \frac{(k+1)(k+2)}{2}$$

find a common denominator!

$$\frac{k(k+1)}{2} + \frac{2(k+1)}{2} =$$

$$\frac{k^2+k+2k+2}{2} =$$

$$\frac{k^2+3k+2}{2} =$$

$$\frac{(k+1)(k+2)}{2} = \checkmark$$

Done!

Example: Use mathematical induction to prove the following formula

$$S_n = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

1. Show the formula is true for n = 1.

$$S_1 = \frac{1^2(1+1)^2}{4} = \frac{4}{4} = 1 \quad \checkmark$$

2. Two steps:

a) Assume that the formula is valid for some integer *k*.

$$S_k = 1^3 + 2^3 + 3^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4}$$

b) Use the assumption to prove that the formula is valid for the next integer, *k* + 1.

(add k+1 on the left side, replace on the right)

$$S_{k+1} = \underbrace{1^3 + 2^3 + 3^3 + \cdots + k^3}_{\frac{k^2(k+1)^2}{4}} + (k+1)^3 = \frac{(k+1)^2(k+1+1)^2}{4}$$

$$\frac{k^2(k+1)^2}{4} + (k+1)^3 = \frac{(k+1)^2(k+2)^2}{4}$$

$$\frac{k^2(k+1)^2 + 4(k+1)^3}{4} =$$

Factor out $(k+1)^2$:

$$\frac{(k+1)^2\left[k^2 + 4(k+1)\right]}{4} =$$

$$\frac{(k+1)^2(k^2+4k+4)}{4} =$$

$$\frac{(k+1)^2(k+2)(k+2)}{4} = \checkmark$$